Section 4

Constructive Mathematics: Foundations and Pilosophy

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Abstract:

Abstract. As opposed to mainstream (or 'classical') mathematics, all logical symbols have computational content in constructive mathematics: Most notably, to state the existence of an object, one must exhibit an algorithm to generate this object. A further fundamental difference is that the trinity foundations, practice, and philosophy are virtually inseparable in constructive mathematics. We shall discuss two related topics in logic and the foundations of mathematics, namely (i) formality and rigour in constructive mathematics and (ii) salient applications of constructive mathematics in computer science, economics, and epistemic logic.

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Titles and Abstracts of the talks

1.
Helmut Schwichtenberg: An introduction to constructive mathematics
There are two ways to prove Euclid's theorem on the existence of infinitely many primes: (i) by deriving a contradiction from the assumption that there are only finitely many, and (ii) by providing a method to construct from finitely many given primes a new one. We call (i) a weak and (ii) a strong existence proof. This distinction has always been in the mind of mathematicians.
However, with the great success of abstract methods introduced around 1900 by Hilbert and others, mathematicians began to ignore the difference. Most of current "classical" mathematics is done that way. However, starting in the early 20th century pioneers like Brouwer, Kolmogorov, Weyl, Bishop and Bridges revived the interest in "constructive" mathematics.
An important aspect is the distinction between computationally relevant (c.r.) and non-computational (n.c.) formulas. The former can be seen (in the spirit of Kolmogorov) as problems asking for a solution ("realizer"), while the latter have been calibrated (via constructive ordinals) in the "extended" Hilbert programme (Schütte, Takeuti, Rathjen). Constructive proofs of c.r. formulas may be used to extract programs, which can be automatically verified if the proof is done formally.
The talk discusses applications for computations with infinite data, in particular real numbers. An important tool is the identification of a c.r. formula with the statement that it has a realizer ("invariance axioms").

2.
Sam Sanders: An introduction to and application of intuitionistic logic.
We will discuss Intuitionistic Epistemic Logic, which is the Artemov-Protopopescu formalisation of knowledge as justified true belief, where true is to be interpreted as constructive truth ([2,3]). We show that the use of intuitionistic logic avoids the usual Gettier problems. We believe this topic provides a highly suitable way of understanding the power of constructive mathematics for philosophers, and a nice application of intuitionistic logic.

3.
Josef Berger: Informal but rigorous foundations for constructive mathematics
Brouwer's school of constructive mathematics (part of his intuitionism) was dwindling after 1950, and it was generally believed that a constructive redevelopment of large parts of mathematics was impossible. Bishop's (in)famous monograph Constructive Analysis ([7]) changed this view overnight. We provide a brief overview of the theory and practice of Constructive Analysis, focusing on its informal-but-rigorous nature. We introduce the closely related field Constructive Reverse Mathematics and discuss some salient applications in economics (based on [5, 6]). We argue that constructive mathematics is more suitable (than classical mathematics) for the development of the exact sciences.

4.
Chuangjie Xu: The computer as referee in mathematics
Among the exact sciences, mathematics is generally viewed as providing (the most) absolute, eternal, and irrevocable knowledge. The actual reality is more complicated: proofs in modern mathematics now cover hundreds (and soon thousands) of pages, with additional computer calculations worth gigabytes of data. Given that no referee (or even team of referees) can reliably judge the correctness of such a proof, how should mathematics as a field proceed?
I will discuss one way forward, namely the use of Martin-Löf's constructive type theory and the associated proof assistants (like Agda and Coq) to formalise and verify the correctness of proofs, as recently proposed by Voevodsky in his homotopy type theory ([17]). I will argue in favour of my own view that the only correct mathematics is formalised mathematics. Time permitting, I will discuss how modern proof assistants manage to come close to absolute reliability and structuralist mathematics.